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Compactness theorems for gradient Ricci solitons

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Abstract

In this paper, we prove a compactness theorem for gradient Ricci solitons. Let (M_α, g_α) be a sequence of compact gradient Ricci solitons of dimension $n \geq 4$, whose curvatures have uniformly bounded $L^{\frac{n}{2}}$ norms, whose Ricci curvatures are uniformly bounded from below, with uniformly lower bounded volume and with uniformly upper bounded diameter; then there must exist a subsequence (M_α, g_α) converging to a compact orbifold (M_∞, g_∞) with finitely many isolated singularities, where g_∞ is a gradient Ricci soliton metric in an orbifold sense.

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1. Introduction

The concept of convergence of Riemannian manifolds was introduced by Gromov [8]. As is now well known, the Cheeger–Gromov convergence theorem [5,8,9,13,17] implies that the space $\mu(\Lambda, \nu, D)$ of compact Riemannian n-manifolds with sectional curvature $|K| \leq \Lambda$, volume $\geq \nu > 0$ and diameter $\leq D$, is precompact in the $C^{1,\alpha}$ topology. There has been increasing interest lately in compactness theorems of Riemannian manifolds under various geometric assumptions [1,2,7,14,20,21]. For instance, in [1] and [14], the authors show that if $\{(M_\alpha, g_\alpha)\}$ is a sequence of Einstein manifolds of dimension n satisfying: $\mathrm{diam}(M_\alpha, g_\alpha) \leq C$; $\int_{M_\alpha} \|\mathrm{Rm}(g_\alpha)\|_{g_\alpha}^{\frac{n}{2}} \, \mathrm{d}V_{g_\alpha} \leq C$; and $\mathrm{Vol}(M_\alpha, g_\alpha) \geq \frac{1}{C}$, where C is a uniform constant, then there

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is a subsequence of $\{(M_\alpha, g_\alpha)\}$ that converges to an Einstein orbifold with finitely many isolated singular points. Also see [20] and [21] for the case of Kähler–Einstein manifolds.

The Ricci flow equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{ij} = -2R_{ij} \tag{1.1}$$

had been introduced by Hamilton in his seminal paper [10]. Natural questions that arise in studying the Ricci flow equation are under what conditions a solution will exist for all times and, if there exists a limit of the solution when we approach infinity, then how we can describe the metric obtained in the limit. In the case of dimension three with positive Ricci curvature and dimension four with positive curvature operator we know (due to Hamilton [10,11]) that the solutions of the Ricci flow in both cases exist for all times, converging to Einstein metrics. In general, we cannot expect to get an Einstein metric in the limit, but we can expect to get solitons in the limit. In brief, a soliton is just a solution to the Ricci flow (1.1) which moves by diffeomorphisms and also shrinks or expands by a factor at the same time. Such a solution is called a homothetic shrinking or expanding Ricci soliton. The equation for a homothetic Ricci soliton is

$$2\lambda g_{ij} - 2R_{ij} = g_{ik}\nabla_j v^k + g_{jk}\nabla_i v^k, \tag{1.2}$$

where λ is the homothetic constant, v is the vector field induced by the 1-parameter family of diffeomorphisms. For $\lambda > 0$ the soliton is shrinking, for $\lambda < 0$ it is expanding, and the case $\lambda = 0$ is a steady Ricci soliton. If the vector field v is the gradient of a function u we say that the soliton is a gradient Ricci soliton; thus

$$\lambda g_{ij} - R_{ij} = \nabla_i \nabla_j u, \tag{1.3}$$

is the gradient Ricci soliton equation. The Einstein metric can be considered as a Ricci soliton when the vector field v is zero.

In this paper, we want to consider the compactness result for Ricci solitons. When the underlying manifolds are closed (compact, without boundary), one can easily check that the steady and expanding Ricci solitons are in fact Einstein metrics. So, we mainly consider the shrinking case. We prove the following theorem.

Theorem 1.1. *Let (M_α, g_α) be a sequence of shrinking gradient Ricci solitons of dimension $n \geq 4$, i.e., satisfying the following equation:*

$$g_\alpha - \mathrm{Ric}(g_\alpha) = \nabla \mathrm{d}u_\alpha, \tag{1.4}$$

such that

(1) $\mathrm{Ric}(g_\alpha) \geq -C_1 g_\alpha$;
(2) $\mathrm{diam}(M_\alpha, g_\alpha) \leq C_2$;
(3) $\mathrm{Vol}(M_\alpha, g_\alpha) \geq C_3$;
(4) $\int_{M_\alpha} |\mathrm{Rm}|^{\frac{n}{2}} \,\mathrm{d}V_{g_\alpha} \leq C_4$;

for some uniform positive constants C_1, C_2, C_3, C_4. Then there is a subsequence (M_α, g_α) converging to (M_∞, g_∞) in the Cheeger–Gromov sense, where M_∞ is an orbifold with finitely many isolated singularities and g_∞ is a Ricci soliton in an orbifold sense.

Further, if n is odd, there are no singular points and (M_∞, g_∞) is a smooth gradient Ricci soliton which is diffeomorphic to M_α, for α sufficiently large. In this case, (M_α, g_α) (sub)converges smoothly to (M_∞, g_∞).

Remark 1.2. An n-dimensional orbifold M_∞ is a topological space satisfying: each point x in M_∞ admits an open neighborhood U_x homeomorphic to B^n/Γ_x, where B^n is the unit disc in R^n and $\Gamma_x \subset O(n)$ is a finite group and those U_x are patched together by smooth transition functions. Any point x with Γ_x trivial is called a regular point of M_∞. In particular, M_∞ is a manifold near such a regular point. Denote by $\mathrm{Reg}(M_\infty)$ the set of all regular points. All other points are singular points of M_∞, i.e. $\mathrm{Sing}(M_\infty) = M_\infty \backslash \mathrm{Reg}(M_\infty)$. We will confine ourselves to the special case where $\mathrm{Sing}(M_\infty)$ consists of isolated points. A Ricci soliton g_∞ in an orbifold sense is just the one on $\mathrm{Reg}(M_\infty)$ such that for each $x \in \mathrm{Sing}(M_\infty)$ if $\pi_x : B^n \to U_x$ is the local uniformization, then there is a diffeomorphism φ of B^n such that $\varphi^*\pi_x^* g_\infty$ can be extended smoothly to a gradient Ricci soliton C^∞-metric on B^n.

In Theorem 1.1, we say that (M_α, g_α) converge to an orbifold (M_∞, g_∞) in the Cheeger–Gromov sense, if for any compact subset $K \subset M_\infty \backslash \mathrm{Sing}(M_\infty)$ there are compact sets $K_\alpha \subset M_\alpha$ and diffeomorphisms $\phi_\alpha : K \to K_\alpha$ such that $\phi_\alpha^* g_\alpha$ converge to g_∞ in C^∞ topology.

Recall the formula of Avez [3] expressing the Euler characteristic $\chi(M)$ of a compact 4-manifold in terms of a curvature integral:

$$\chi(M) = \frac{1}{8\pi^2} \int_M |\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + R^2, \tag{1.5}$$

where R is the scalar curvature. Clearly, a bound on $\int_M |\mathrm{Ric}|^2$ and the second Betti number $b_2(M)$ implies a bound on $\int_M |\mathrm{Rm}|^2$. The Bishop comparison theorem implies there is a upper bound of volume imposed by the lower bound of Ricci curvature and the upper bound of diameter. On the other hand, in Section 2, we will prove that when g is a gradient Ricci soliton, the lower bound of Ricci curvature, the upper bound of diameter, and the lower bound of volume imply an upper bound of scalar curvature, then we have an upper bound of Ricci curvature. So, for Ricci solitons, lower bounds of Ricci curvature and volume, an upper bound of diameter, and a bound of $b_2(M)$ imply a bound on $\int_M |\mathrm{Rm}|^2$. We have the following corollary.

Corollary 1.3. *Let (M_α, g_α) be a sequence of shrinking gradient Ricci solitons of dimension* 4, *such that*

(1) $\mathrm{Ric}(g_\alpha) \geq -C_1 g_\alpha$;
(2) $\mathrm{diam}(M_\alpha, g_\alpha) \leq C_2$;
(3) $\mathrm{Vol}(M_\alpha, g_\alpha) \geq C_3$;
(4) $b_2(M_\alpha) \leq C_4$,

for some uniform positive constants C_1, C_2, C_3, C_4. Then there is a subsequence (M_α, g_α) converging to (M_∞, g_∞) in the Cheeger–Gromov sense, where M_∞ is an orbifold with finitely many isolated singularities and g_∞ is a Ricci soliton in an orbifold sense.

More recently, Cao and Sesum [6] proved a compactness result for the Kähler Ricci solitons, where the upper bound of diameter can be replaced by a uniform lower bound of Perelman's functional $\mu(g, \frac{1}{2})$ [15]. They point out that by the same proof as in [16] they can show that, in the Kähler Ricci solitons case, the uniform lower bound of Ricci curvature, the lower bound of Perelman's functional $\mu(g, \frac{1}{2})$, and the Euclidean volume growth imply a uniform bound of the diameter. In Section 2, we will show that, for a sequence of gradient Ricci solitons, uniform lower bounds of Ricci curvature and volume, and a uniform bound of diameter will give a uniform bound of Perelman's functional $\mu(g, \frac{1}{2})$.

The key analytic tool in this paper is Uhlenbeck's [22] Yang–Mills estimate for curvatures of Yang–Mills connections. The method for proving Theorem 1.1 is similar to that in Anderson's paper [1] for the Einstein case. The main point is obtaining the ϵ-regularity estimate for the Ricci soliton which says that smallness of the $L^{\frac{n}{2}}$ norm of curvature implies a pointwise bound on the curvature. Moreover, unlike in the Einstein case, we should obtain a C^1 bound for potential functions u_α. It is well known [23] that lower bounds for the Ricci curvature and volume and an upper bound on the diameter give a lower bound for the Sobolev constant C_s of compact manifold M,

$$\|u\|_{\frac{2n}{n-2}} \leq \frac{1}{C_s}\|\mathrm{d}u\|_2 + \mathrm{Vol}(M)^{-\frac{2}{n}}\|u\|_2, \tag{1.6}$$

for any Lipschitz function u on M. By the above Sobolev inequality, we can use the Moser's iteration argument to obtain the above estimates.

The organization of this paper is as follows. In Section 2, we deduce some estimates; in particular, we obtain the C^1 estimates for functions u_α, and a uniform bound for the Ricci curvature and Perelman's function. In Section 3, we obtain the ϵ-regularity estimate for the Ricci soliton. In Sections 4 and 5, we give the proof of Theorem 1.1.

2. Preliminary results

Let M be a compact manifold without boundary, and g be a gradient Ricci soliton, i.e. one that satisfies formula (1.3); here we assume that u satisfies

$$(2\pi)^{-\frac{n}{2}}\int_M \mathrm{e}^{-u}\,\mathrm{d}V_g = 1 \tag{2.1}$$

and $\mathrm{Ric}(g) \geq -C_1 g$; $\mathrm{diam}(M, g) \leq C_2$; $\mathrm{Vol}(M, g) \geq C_3 > 0$.

From formula (2.1), we have

$$\inf_{x\in M} u \leq \ln \mathrm{Vol}(M, g) - \frac{n}{2}\ln(2\pi). \tag{2.2}$$

On the other hand, the Bishop comparison theorem implies there is an upper bound of volume imposed by a lower bound of Ricci curvature and an upper bound of the diameter. So, there exists a constant C_5 depending only on C_1 and C_2 such that

$$\inf_{x\in M} u \leq C_5. \tag{2.3}$$

Let $f = \mathrm{e}^{-\frac{u}{2}}$; then we have

$$\begin{aligned}\triangle f^2 &= \mathrm{e}^{-u}|\nabla u|^2 - \mathrm{e}^{-u}\triangle u\\ &= \mathrm{e}^{-u}|\nabla u|^2 + \mathrm{e}^{-u}(R - n\lambda)\\ &\geq 4|\nabla f|^2 - nf^2(C_1+\lambda).\end{aligned} \tag{2.4}$$

From the above Bochner-type inequality and the Sobolev inequality (1.6), using Moser's iteration argument ([24], Proposition 2.2), we have the following mean value inequality:

$$\sup_{x\in M} f \leq C_6\left(\int_M \mathrm{e}^{-u}\mathrm{d}V_g\right)^{\frac{1}{2}}, \tag{2.5}$$

where C_6 depends only on C_1, C_2 and C_3. So, we obtain a lower bound of u, i.e. there is a constant C_7 depending only on C_1, C_2 and C_3 such that

$$\inf_M u \geq -C_7. \tag{2.6}$$

From the Ricci soliton equation and the lower bound of Ricci curvature, we have

$$\nabla \mathrm{d}u \leq (C_1 + \lambda)g. \tag{2.7}$$

Let $P, Q \in M$ be such that $u(P) = \inf_{x\in M} u$, $u(Q) = \sup_{x\in M} u$, and let $\gamma : [0, d] \to M$ be a minimizing geodesic connecting P and Q, i.e. $\gamma(0) = P$, $\gamma(d) = Q$; here $d = \mathrm{dist}(P, Q)$. We have

$$\begin{aligned}\frac{\mathrm{d}u(\gamma(t))}{\mathrm{d}t} &= \langle \nabla u, \gamma'\rangle_{\gamma(t)} - \langle \nabla u, \gamma'\rangle_P \\ &= \int_0^t \frac{\partial}{\partial s}(\langle \nabla u, \gamma'\rangle_{\gamma(s)})\,\mathrm{d}s \\ &= \int_0^t \nabla_{\gamma'(s)}\langle \nabla u, \gamma'\rangle\,\mathrm{d}s \\ &= \int_0^t (\nabla \mathrm{d}u)(\gamma', \gamma')\,\mathrm{d}s \\ &\leq (C_1 + \lambda)t,\end{aligned}$$

and

$$u(Q) - u(P) = \int_0^d \frac{\mathrm{d}u(\gamma(t))}{\mathrm{d}t}\,\mathrm{d}t \leq \int_0^d (C_1 + \lambda)t\mathrm{d}t = \frac{1}{2}(C_1 + \lambda)d^2.$$

From the above inequality and (2.3), we know that there exists a constant C_8 depending only on C_1, C_2, and C_3, such that

$$\sup_{x\in M} u \leq C_8. \tag{2.8}$$

Next, we want to obtain the estimate of $|\nabla u|$. From Eq. (1.3), we have

$$\nabla_i R_{jk} - \nabla_j R_{ik} = -R_{ijkl}\nabla_l u. \tag{2.9}$$

Taking a trace on j and k, and using the second Bianchi identity we have

$$\nabla_i R - 2R_{ij}\nabla_j u = 0, \tag{2.10}$$

and

$$\nabla_i(|\nabla u|^2 + R - 2\lambda u) = 0. \tag{2.11}$$

So, there is a constant C_9 such that

$$|\nabla u|^2 + R - 2\lambda u = C_9. \tag{2.12}$$

As above, we let $P \in M$ be the minimum point of u; then $|\nabla u|(P) = 0$, $\triangle u(P) \geq 0$, and $R(P) = n\lambda - \triangle u(P) \leq n\lambda$. We have

$$C_9 = |\nabla u|^2(P) + R(P) - 2\lambda u(P) \leq n\lambda - 2\lambda u(P). \tag{2.13}$$

From (2.12), we have

$$|\nabla u|^2 = -R + 2\lambda u + C_9 \le n\lambda + 2\lambda\left(u - \inf_{x\in M} u\right) - R. \tag{2.14}$$

When the constant $\lambda \le 0$, from the above inequality, we have

$$R \le R + |\nabla u|^2 \le n\lambda, \tag{2.15}$$

and then $\triangle u = n\lambda - R \ge 0$. Since manifold M is compact, u must be a constant. So we have the following proposition.

Proposition 2.1. *Let g be a steady or expanding gradient soliton over compact manifold M; then g must be a Einstein metric.*

When λ is a positive constant, from the estimate (2.8) we have

$$|\nabla u|^2 \le n\lambda + 2\lambda\left(\sup_{x\in M} u - \inf_{x\in M} u\right) - R \le C_{10}, \tag{2.16}$$

where C_{10} is a constant depending only on C_1, C_2, C_3 and λ.

Let (M_α, g_α) be a sequence of shrinking Ricci solitons satisfying conditions (1), (2) and (3) in Theorem 1.1. From (2.6), (2.8) and (2.16), we obtain a uniform C^1-bound of u_α; from (2.8) and (2.14); we also obtain a uniform upper bound of scalar curvature, i.e. we obtain the following lemma.

Lemma 2.2. *Let (M_α, g_α) be a sequence of shrinking Ricci solitons ($\lambda = 1$) satisfying conditions* (1), (2) *and* (3) *in* Theorem 1.1, *and u_α satisfying the constraint* (2.1); *then there are positive constants C_{11}, C_{12} depending only on C_1, C_2 and C_3 such that*

$$|u_\alpha|_{C^1} \le C_{11} \tag{2.17}$$

and

$$R(g_\alpha) \le C_{12}. \tag{2.18}$$

In the next part of this section, we will give a uniform bound of Perelman's functional $\mu(g, \frac{1}{2})$ [15] for a sequence of shrinking Ricci solitons (M_α, g_α) satisfying conditions (1), (2) and (3) in Theorem 1.1. In [15], Perelman has introduced a functional satisfying

$$W(g, \varphi, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M \mathrm{e}^{-\varphi}[\tau(R + |\nabla\varphi|^2) + f - n]\,\mathrm{d}V_g, \tag{2.19}$$

under the constraint

$$(4\pi\tau)^{-\frac{n}{2}} \int_M \mathrm{e}^{-\varphi}\,\mathrm{d}V_g = 1. \tag{2.20}$$

Then he defined the functional

$$\mu(g, \tau) = \inf W(g, \cdot, \tau), \tag{2.21}$$

where $\tau > 0$, and inf is taken over all functions satisfying the constraint (2.20).

Lemma 2.3. *If (M, g) is a shrinking gradient Ricci soliton, i.e.*

$$g - \mathrm{Ric}(g) = \nabla \mathrm{d}u,$$

where u satisfies the constraint (2.1), then u is a minimizer of Perelman's functional W with respect to metric g and $\tau = \frac{1}{2}$.

Proof. Let $\psi(t)$ be the 1-parameter family of diffeomorphisms that come from the vector field ∇u, and let $g(t) = \psi^* g$; then $g(t)$ satisfies the following Ricci flow equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = -2\mathrm{Ric}(g(t)) + 2g(t). \tag{2.22}$$

In order to use Perelman's monotonicity formula [15], we scale the metric via $\tilde{g}(s) = C(s)g(t(s))$, where $C(s) = 1 - 2s$, $t(s) = -\frac{1}{2}\ln(1 - 2s)$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde{g}(s) = -2\mathrm{Ric}(\tilde{g}(s)), \tag{2.23}$$

and

$$\mu\left(g(t), \frac{1}{2}\right) = \mu\left(\tilde{g}(s(t)), \frac{1}{2} - s(t)\right). \tag{2.24}$$

Let $\varphi(0)$ be a minimizer of W with respect to metric $g(0) = g$ and $\tau = \frac{1}{2}$. Then function $\varphi(t) = \psi^*\varphi(0)$ is a minimizer of W with respect to metric $g(t)$ since

$$\begin{aligned} \mu\left(g(t), \frac{1}{2}\right) &\le W\left(g(t), \varphi(t), \frac{1}{2}\right) = W\left(g(0), \varphi(0), \frac{1}{2}\right) \\ &= \mu\left(g(0), \frac{1}{2}\right) \le \mu\left(g(t), \frac{1}{2}\right), \end{aligned} \tag{2.25}$$

where the last inequality comes from Perelman's monotonicity formula for $\mu(\tilde{g}(s), \frac{1}{2} - s)$. Therefore, we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} W\left(g(t), \varphi(t), \frac{1}{2}\right) \\ &= (2\pi)^{\frac{1}{2}} \int_M \mathrm{e}^{-\varphi(t)} |R_{ij} + \varphi_{ij} - g_{ij}|^2 \,\mathrm{d}V_t, \end{aligned} \tag{2.26}$$

which implies $\triangle\varphi(t) = n - R(t) = \triangle(u \circ \psi(t))$. Since M is compact and both functions satisfy the constraint (2.1), we have $\varphi(t) = u \circ \psi(t)$. □

For our sequence of shrinking gradient Ricci solitons (M_α, g_α), the previous lemma tells us that every u_α is a minimizer of $W(g_\alpha, \cdot, \frac{1}{2})$ and therefore satisfies [15]

$$\triangle u_\alpha - \frac{1}{2}|\nabla u_\alpha|^2 + \frac{1}{2}R(g_\alpha) + u_\alpha - n = \mu\left(g_\alpha, \frac{1}{2}\right). \tag{2.27}$$

By $\triangle u_\alpha = n - R(g_\alpha)$ and Lemma 2.2, we obtain a uniform bound of $\mu(g_\alpha, \frac{1}{2})$.

Proposition 2.4. *Let (M_α, g_α) be a sequence of shrinking Ricci solitons ($\lambda = 1$) satisfying conditions* (1), (2) *and* (3) *in Theorem* 1.1, *and let u_α satisfy the constraint (2.1); then there is a* *constant C_{13} depending only on C_1, C_2 and C_3 such that*

$$\left|\mu\left(g_\alpha, \frac{1}{2}\right)\right| \le C_{13}. \tag{2.28}$$

Remark 2.5. From Lemma 2.2, we get scalar curvature bounds for Ricci solitons satisfying conditions (1), (2) and (3) in Theorem 1.1. Since Ricci curvature have a lower bound, we can also get an upper bound for the Ricci curvature. If the dimension $n = 3$, we know that Ricci curvature bounds imply Riemannian curvature bounds. On the other hand, by Shi's estimates [18] we can get uniform bounds for higher derivatives of Riemannian curvature; then using the Gromov–Cheeger compactness theorem, we can easily get the following compactness theorem for Ricci solitons.

Proposition 2.6. *Let (M_α, g_α) be a sequence of shrinking gradient Ricci solitons of dimension $n = 3$, such that*

(1) $\mathrm{Ric}(g_\alpha) \ge -C_1 g_\alpha$;
(2) $\mathrm{diam}(M_\alpha, g_\alpha) \le C_2$;
(3) $\mathrm{Vol}(M_\alpha, g_\alpha) \ge C_3 > 0$;

for some uniform constants C_1, C_2, C_3. Then there is a subsequence (M_α, g_α) converging to (M_∞, g_∞) in C^∞ topology, and (M_∞, g_∞) is a smooth gradient Ricci soliton.

3. ϵ-Regularity for Ricci solitons

Let (M, g) be a shrinking gradient Ricci soliton. Choose a normal coordinate system on the point considered; by direct calculation, we have

$$\begin{aligned}
\triangle R_{ijkl} &= \nabla_m \nabla_m R_{ijkl} \\
&= -\nabla_m \nabla_k R_{ijlm} - \nabla_m \nabla_l R_{ijmk} \\
&= -\nabla_k \nabla_m R_{ijlm} - \nabla_l \nabla_m R_{ijmk} + Q(\mathrm{Rm})_{ijkl} \\
&= \nabla_k \nabla_m R_{mlij} - \nabla_m \nabla_l R_{mkij} + Q(\mathrm{Rm})_{ijkl} \\
&= \nabla_k \nabla_i R_{lj} - \nabla_k \nabla_j R_{li} - \nabla_l \nabla_i R_{kj} + \nabla_l \nabla_j R_{ki} + Q(\mathrm{Rm})_{ijkl} \\
&= \nabla_k (R_{mlij} \nabla_m u) - \nabla_l (R_{mkij} \nabla_m u) + Q(\mathrm{Rm})_{ijkl} \\
&= \nabla_k R_{mlij} \nabla_m u + R_{mlij} \nabla_k \nabla_m u - \nabla_l R_{mkij} \nabla_m u \\
&\quad - R_{mkij} \nabla_l \nabla_m u + Q(\mathrm{Rm})_{ijkl},
\end{aligned} \tag{3.1}$$

where we have used the second Bianchi identity, the Ricci identity, and formula (2.9), $Q(\mathrm{Rm})$ denotes a quadratic express in the curvature tensor. In shorthand form, we write the above identity as

$$\triangle \mathrm{Rm} = \nabla \mathrm{Rm} * \nabla u + \mathrm{Rm} * g + \mathrm{Rm} * \mathrm{Ric} + \mathrm{Rm} * \mathrm{Rm}. \tag{3.2}$$

Then, we have

$$\begin{aligned}
\triangle |\mathrm{Rm}|^2 &= 2|\nabla \mathrm{Rm}|^2 + 2\langle \triangle \mathrm{Rm}, \mathrm{Rm}\rangle \\
&\ge 2|\nabla \mathrm{Rm}|^2 - 4|\nabla \mathrm{Rm}||\nabla u||\mathrm{Rm}| - C_{14}|\mathrm{Rm}|^2 - C_{14}|\mathrm{Rm}|^3,
\end{aligned} \tag{3.3}$$

where C_{14} is a positive constant depending only on dimension n. By using the estimate (2.17) and the Kato inequality, we get

$$\triangle|\mathrm{Rm}|^2 \geq (2-\theta)|\nabla|\mathrm{Rm}||^2 - C(\theta)|\mathrm{Rm}|^2 - C_{14}|\mathrm{Rm}|^3. \tag{3.4}$$

Next, we use Moser's iteration argument to deduce the following mean value inequality.

Lemma 3.1. *Let (M, g) be a compact Riemannian manifold, and f be a Lipschitz function satisfying*

$$f\triangle f \geq -\theta_1|\nabla f|^2 - \theta_2 f^2 - \theta_3 f^3, \tag{3.5}$$

in the weak sense. Suppose that $\theta_1 \leq \frac{1}{4}$; then there exists a constant ϵ depending only on the dimension of M, θ_3 and the lower bound of the Sobolev constant C_s so that if

$$\int_{B_P(2r)} f^{\frac{n}{2}} \mathrm{d}v_g \leq \epsilon, \tag{3.6}$$

then

$$\sup_{B_P(\frac{r}{2})} f \leq C_* \left(1+\frac{1}{r^2}\right)\left(\int_{B_P(r)} f^{\frac{n}{2}}\,\mathrm{d}v_g\right)^{\frac{2}{n}}, \tag{3.7}$$

where C_* depends only on the dimension of M, θ_2, θ_3, the lower bound of $\mathrm{Vol}(M)$ *and the Sobolev constant C_s.*

Proof. Multiplying $\eta^2 f^{q-1}$ by (3.5), and integrating, yields

$$\begin{aligned}
&\frac{4\theta_1}{q^2}\int_M \eta^2|\nabla f^{\frac{q}{2}}|^2 + \theta_2\int_M \eta^2 f^q + \theta_3\int_M \eta^2 f^{q+1} \geq -\int_M \eta^2 f^{q-1}\triangle f\\
&\quad = \frac{4}{q}\int_M \eta f^{\frac{q}{2}}\langle\nabla\eta, \nabla f^{\frac{q}{2}}\rangle + \frac{4(q-1)}{q^2}\int_M \eta^2|\nabla f^{\frac{q}{2}}|^2\\
&\quad \geq -\frac{2}{q-1}\int_M f^q|\nabla\eta|^2 + \frac{2(q-1)}{q^2}\int_M \eta^2|\nabla f^{\frac{q}{2}}|^2,
\end{aligned} \tag{3.8}$$

where $q \geq 2$ and η is a nonnegative cut-off function that we will choose later. If we suppose that $\theta_1 \leq \frac{1}{4}$, from the above inequality, we have

$$\frac{(q-1)}{q^2}\int_M \eta^2|\nabla f^{\frac{q}{2}}|^2 \leq \frac{2}{q-1}\int_M f^q|\nabla\eta|^2 + \theta_2\int_M \eta^2 f^q + \theta_3\int_M \eta^2 f^{q+1}. \tag{3.9}$$

Using the Sobolev inequality (1.6), and letting $\mu = \frac{n}{n-2}$, we obtain

$$\begin{aligned}
\left\{\int_M (\eta f^{\frac{q}{2}})^{2\mu}\right\}^{\frac{1}{\mu}} &\leq \frac{2}{C_s^2}\int_M |\nabla(\eta f^{\frac{q}{2}})|^2 + 2\mathrm{Vol}(M)^{-\frac{4}{n}}\int_M \eta^2 f^q\\
&\leq \frac{2}{C_s^2}\left\{\int_M \eta^2|\nabla f^{\frac{q}{2}}|^2 + \int_M f^q|\nabla\eta|^2\right\} + 2\mathrm{Vol}(M)^{-\frac{4}{n}}\int_M \eta^2 f^q\\
&\leq \frac{2\theta_3 q^2}{C_s^2(q-1)}\int_M \eta^2 f^{q+1} + \frac{6q^2}{C_s^2(q-1)^2}\int_M f^q|\nabla\eta|^2\\
&\quad + \left(\frac{2\theta_2 q^2}{C_s^2(q-1)} + 2\mathrm{Vol}(M)^{-\frac{4}{n}}\right)\int_M \eta^2 f^q.
\end{aligned} \tag{3.10}$$

By the Hölder inequality, we have

$$\int_M \eta^2 f^{q+1} \leq \left(\int_{\mathrm{Supp}\eta} f^{\frac{n}{2}}\right)^{\frac{2}{n}} \left\{\int_M (\eta f^{\frac{q}{2}})^{2\mu}\right\}^{\frac{1}{\mu}}. \tag{3.11}$$

Take $\epsilon < \{\frac{(n-2)C_s^2}{2\theta_3 n^2}\}$. Let $q = \frac{n}{2}$ and let η be a cut-off function with compact support in $B_P(2r)$, equal to 1 on $B_P(r)$ and such that $|\nabla\eta| \leq \frac{2}{r}$; from the above inequalities, we get

$$\left\{\int_{B_P(r)} f^{\frac{n}{2}\mu}\right\}^{\frac{1}{\mu}} \leq \left(\frac{48n^2}{C_s^2(n-2)^2}\frac{1}{r^2} + \frac{4n^2}{C_s^2(n-2)^2}\theta_2 + 4\mathrm{Vol}(M)^{-\frac{4}{n}}\right)\int_{B_P(2r)} f^{\frac{n}{2}}. \tag{3.12}$$

In the next part of the proof, we choose cut-off functions η with compact support in $B_P(r)$, and equal to 1 on $B_P(\frac{r}{2})$. Using the Hölder inequality again, we get

$$\int_M \eta^2 f^{q+1} \leq \left\{\int_{B_P(r)} f^{\frac{n}{2}\mu}\right\}^{\frac{2}{n\mu}} \left(\int_M (\eta f^{\frac{q}{2}})^{2\nu}\right)^{\frac{1}{\nu}} \tag{3.13}$$

where $\nu = \frac{\frac{n}{2}\mu}{\frac{n}{2}\mu - 1} = \frac{n^2}{n^2-2n+4}$. Using the Young inequality, we have

$$\left(\int_M (\eta f^{\frac{q}{2}})^{2\nu}\right)^{\frac{1}{\nu}} \leq \delta\left(\int_M (\eta f^{\frac{q}{2}})^{2\mu}\right)^{\frac{1}{\mu}} + C(n)\delta^{-\frac{n-2}{2}}\int_M \eta^2 f^q, \tag{3.14}$$

for small δ.

Setting $\delta = \frac{1}{2}\{\int_{B_P(r)} f^{\frac{n}{2}\mu}\}^{\frac{2}{n\mu}}\frac{C_s^2(q-1)}{2\theta_3 q^2}$, from (3.10), (3.13) and (3.14) we have

$$\left\{\int_M (\eta^2 f^q)^{\mu}\right\}^{\frac{1}{\mu}} \leq C_{15} q^n \int_M \left(\left(1+\frac{1}{r^2}\right)\eta^2 + |\nabla\eta|^2\right) f^q, \tag{3.15}$$

where C_{15} is a positive constant depending only on θ_2, θ_3, the dimension of M, the lower bound of $\mathrm{Vol}(M)$ and the Sobolev constant.

Set $\frac{r}{2} \leq r_2 < r_1 \leq r$, and let $\eta \in C_0^\infty(B_P(r_1))$ be the cut-off function with the property that $\eta = 1$ in $B_P(r_2)$ and $|\nabla\eta| \leq \frac{2}{r_1 - r_2}$. From (3.15), we have

$$\left(\int_{B_P(r_2)} f^{q\mu}\right)^{\frac{1}{\mu}} \leq 4C_{15} q^n \left(1 + \frac{1}{r^2} + \frac{1}{(r_1-r_2)^2}\right)\int_{B_P(r_1)} f^q. \tag{3.16}$$

Let $R_i = \frac{r}{2} + \frac{r}{2}2^{-i}$, $q_i = \frac{n}{2}\mu^i$; applying (3.16) to $r_1 = R_i$, $r_2 = R_{i+1}$, $q = q_i$, we have

$$\left(\int_{B_P(R_{i+1})} f^{\frac{n}{2}\mu^{i+1}}\right)^{\mu^{-(i+1)}} \leq \left(64C_{15}\left(1+\frac{1}{r^2}\right)\frac{n}{2}\right)^{\mu^{-i}} (2\mu)^{n\cdot i\mu^{-i}} \left(\int_{R_i} f^{\frac{n}{2}\mu^i}\right)^{\mu^{-i}}. \tag{3.17}$$

Observe that $\lim_{i\to\infty} R_i = \frac{r}{2}$, and iterating the above inequality, we conclude that

$$\sup_{B_P(\frac{r}{2})} f^{\frac{n}{2}} \leq C_{16}\left(1+\frac{1}{r^2}\right)^{\frac{n}{2}} \int_{B_p(r)} f^{\frac{n}{2}}. \quad \square \tag{3.18}$$

Let $\theta = \frac{1}{2}$ in the formula (3.4); then the norm of Riemannian curvature |Rm| of shrinking gradient Ricci solitons must satisfy the Bochner-type inequality (3.5). From Lemma 3.1, we obtain the ϵ regularity estimates for shrinking gradient Ricci solitons.

Theorem 3.2. *Let* (M_α, g_α) *be a sequence of compact gradient Ricci solitons satisfying the conditions* (1), (2) *and* (3) *in the Theorem* 1.1. *Then there exist constants* C_{17} *and* ϵ *depending only on* C_1, C_2, C_3 *such that if*

$$\int_{B_P^\alpha(2r)} |\mathrm{Rm}(g_\alpha)|^{\frac{n}{2}} < \epsilon, \tag{3.19}$$

then

$$\sup_{B_P^\alpha(\frac{r}{2})} |\mathrm{Rm}(g_\alpha)| \leq C_{17}\left(1+\frac{1}{r^2}\right)\left(\int_{B_P^\alpha(r)} |\mathrm{Rm}(g_\alpha)|^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{3.20}$$

4. The proof of Theorem 1.1

In this section, firstly, we will show that we can extract a subsequence of Ricci solitons (M_α, g_α) which satisfy conditions (1), (2), (3) and (4) in Theorem 1.1, so that it converges to an orbifold in a topological sense. This relies on the works by Anderson [1,2].

Let (M, g) be a Riemannian manifold, and let h_M be the isoperimetric constant given by

$$h_M = \inf_S \frac{(\mathrm{Vol}(M))^n}{[\min(\mathrm{Vol}(M_1), \mathrm{Vol}(M_2))]^{n-1}}, \tag{4.1}$$

where S varies over all closed hypersurfaces of M such that $M\backslash S = M_1 \cup M_2$. Croke shows that h_M is bounded below by a constant depending only on lower bounds for Ricci curvature and volume, and an upper bound on the diameter. In particular, if $B_x(r)$ is a geodesic ball of radius r about $x \in M$ and $S_x(r) = \partial B_x(r)$, $v(r) = \mathrm{Vol}(B_x(r))$, then it follows that $(v'(r))^n \geq h_M v(r)^{n-1}$, for $v(r) < \frac{1}{2}\mathrm{Vol}(M)$; integrating this inequality, one obtains $v(r) \geq n^{-n} h_M r^n$. On the other hand, from the Bishop volume comparison theorem, we know that there must exist a positive constant C_{18} depending only on the lower bounds of Ricci curvature and volume such that

$$\mathrm{Vol}(B_x(r)) < \frac{1}{2}\mathrm{Vol}(M), \quad \text{when } r < C_{18}. \tag{4.2}$$

So, we have

$$\mathrm{Vol}(B_x(r)) \geq C_{19} r^n, \tag{4.3}$$

for $r < C_{18}$, where C_{19} depends only on a lower bound for the isoperimetric constant. From [23], the volume noncollapsing condition (4.3) and a lower bound for the isoperimetric constant imply the following Sobolev inequality:

$$\left(\int_{B_x(r)} f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{1}{C_s'}\int_{B_x(r)} |\nabla f|^2, \tag{4.4}$$

for every Lipschitz function f with compact support in $B_x(r)$ and $r \leq C_{18}$. In fact, an upper bound on the diameter, a lower bound of the Ricci curvature and the volume noncollapsing condition imply a lower bound on the Sobolev constant C_s'.

Let ϵ be the constant of Theorem 3.2 which is determined by the bounds $\mathrm{Ric}(g_\alpha) \geq -C_1 g_\alpha$, $\mathrm{diam}(M_\alpha, g_\alpha) \leq C_2$, $\mathrm{Vol}(M_\alpha, g_\alpha) \leq C_3$ on (M_α, g_α). We fix $0 < r < C_{18}$ and let $\{x_k^\alpha\}$ be a maximal $\frac{r}{8}$ separated set in (M_α, g_α). Thus the geodesic balls $B_{x_k}^\alpha(\frac{r}{16})$ are disjoint, and the balls $B_{x_k}^\alpha(\frac{r}{4})$ form a cover of M_α. We let

$$G_\alpha^r = \cup \left\{ B_{x_k}^\alpha \left(\frac{r}{4}\right) : \int_{B_{x_k}^\alpha(2r)} |\mathrm{Rm}(g_\alpha)|^{\frac{n}{2}} < \epsilon \right\},$$

and

$$B_\alpha^r = \cup \left\{ B_{x_k}^\alpha \left(\frac{r}{4}\right) : \int_{B_{x_k}^\alpha(2r)} |\mathrm{Rm}(g_\alpha)|^{\frac{n}{2}} \geq \epsilon \right\}.$$

Then $M_\alpha = G_\alpha^r \cup B_\alpha^r$. From the volume noncollapsing condition (4.3) and Bishop–Gromov volume estimates, we have a bound on the number of bad balls Q_α^r in B_α^r independent of α and r, namely,

$$Q_\alpha^r \leq C_{20}, \tag{4.5}$$

where C_{20} is positive constant depending only on the constants C_1, C_2, C_3, C_4 which are given in Theorem 1.1.

Since Ricci solitons are the solutions for Ricci flow, Shi's curvature estimates apply and therefore, by the estimates (3.20),

$$\sup_{G_\alpha^r} |\nabla^k \mathrm{Rm}(g_\alpha)| \leq \frac{C_{21}}{r^{k+2}}, \tag{4.6}$$

where C_{21} depends only on C_1, C_2, C_3. We also obtain

$$\sup_{G_\alpha^r} |\nabla^k u_\alpha| \leq C_{22}(k), \tag{4.7}$$

where $C_{22}(k)$ is a constant depending only on k, r, C_1, C_2, C_3.

From the volume noncollapsing condition (4.3), small curvature estimates (Theorem 3.2), and Shi's curvature estimates (4.6), following Section 5 in [1], we can show that there is a subsequence of (M_α, g_α) that converges to (M_∞, g_∞) in the Hausdorff topology, and $M_\infty = G \cup \{P_i\}_1^Q$ is a complete length space with a length function g_∞, which restricts to a smooth gradient Ricci soliton on G satisfying

$$g_\infty - \mathrm{Ric}(g_\infty) = \nabla \mathrm{d} u_\infty, \tag{4.8}$$

where u_∞ is a C^∞ limit of u_α away from singular points. The points $\{P_i\}_1^Q$ are called the curvature singularities of M_∞, and the convergence is in C^∞-topology outside the singularities. Then, in a similar way to in Section 5 in [1], we can check that M_∞ has the structure of an orbifold with a finite number of curvature singularity points, each having a punctured neighborhood which is diffeomorphic to a punctured cone on a spherical space form, and metric g_∞ has a C^0 extension over every singularity point. So, we have proved the following proposition.

Proposition 4.1. *Let (M_α, g_α) be a sequence of compact gradient Ricci solitons satisfying the conditions* (1), (2), (3) *and* (4) *in* Theorem 1.1. *There is a subsequence such that (M_α, g_α) converges to a compact orbifold (M_∞, g_∞) with finitely many singularities. Convergence is* *in C^∞ topology outside those singularity points; g_∞ is a smooth Ricci soliton outside those singularities and has a C^0 extension over every singularity point.*

To finish the proof of Theorem 1.1 we still need to show that the limit metric g_∞ on G can be extended to an orbifold metric on M_∞. More precisely, in an orbifold lifting around singular points, in an appropriate gauge, the gradient Ricci soliton equation of g_∞ can be smoothly extended over the origin in a ball in R^n. At this stage, the regularity theory is not sufficient to imply that g_∞ is smooth. However, by Fatou's lemma, we know that

$$\int_{M_\infty} |\mathrm{Rm}(g_\infty)|^{\frac{n}{2}}\, \mathrm{d}v_\infty < \infty. \tag{4.9}$$

From the above inequality, we can obtain an upper bound for the norm of the curvature tensor $\mathrm{Rm}(g_\infty)$ of the limit metric g_∞.

Lemma 4.2. $|\mathrm{Rm}(g_\infty)|_\infty$ *is bounded uniformly on* $M_\infty \backslash \{P_i\}_1^Q$.

We leave the proof of Lemma 4.2 to the next section. From the above, we know that each singular point $P \in M_\infty$ has a neighborhood that is covered by a punctured ball $B^n(r)\backslash\{0\} \in R^n$. Our goal is to show that there exists a diffeomorphism ϕ of $B^n(r)\backslash\{0\}$ such that $\phi^*\pi^*(g_\infty)$ extends to a smooth metric on $B^n(r)$, where π is the covering map.

Using Lemma 4.2, and harmonic coordinates constructed in [12], in the same way as in [4, Theorem 5.1], we can show that if r is sufficiently small, there is a diffeomorphism ϕ of $B^n(r)\backslash\{0\}$ such that ϕ extends to a homeomorphism of $B^n(r)$ and satisfies

$$\begin{aligned} &(g_\infty)_{ij}(x) - \delta_{ij} = O(|x|^2), \\ &\partial_k (g_\infty)_{ij}(x) = O(|x|), \end{aligned} \tag{4.10}$$

where we also denote the pulled back metric $\phi^*\pi^*(g_\infty)$ as g_∞ for simplicity. This means that there are some coordinates in a covering of a singular point of M_∞ in which g_∞ extends to a $C^{1,1}$-metric.

For our sequence of shrinking gradient Ricci solitons (M_α, g_α), Lemma 2.3 tells us that every potential function u_α is a minimizer of $W(g_\alpha, \cdot, \frac{1}{2})$ and therefore satisfies

$$\triangle u_\alpha - \frac{1}{2}|\nabla u_\alpha|^2 + \frac{1}{2}R(g_\alpha) + u_\alpha - n = \mu\left(g_\alpha, \frac{1}{2}\right).$$

By $\triangle u_\alpha = n - R(g_\alpha)$, we have

$$\triangle u_\alpha = |\nabla u_\alpha|^2 - 2u_\alpha + n + 2\mu\left(g_\alpha, \frac{1}{2}\right). \tag{4.11}$$

Proposition 2.4 tells us that those $\mu(g_\alpha, \frac{1}{2})$ are bounded uniformly. So we can extract a subsequence of a sequence of converging metrics g_α such that

$$\lim_{\alpha\to\infty} \mu\left(g_\alpha, \frac{1}{2}\right) = \mu_\infty. \tag{4.12}$$

Letting $\alpha \to \infty$ in (4.11) we get

$$\triangle u_\infty = |\nabla u_\infty|^2 - 2u_\infty + n + 2\mu_\infty, \tag{4.13}$$

away from those singular points P_i. On the other hand, from Lemma 2.2 and the soliton equation, we have

$$\sup_{M_\infty \setminus \{P_i\}_1^Q} |u_\infty|_{C^2} \le C_*, \tag{4.14}$$

for some uniform constant C_*. So, it is not hard to conclude that ∇u_∞ extends to the origin in the covering ball $B^n(r)$. Moreover, $u_\infty \in C^{1,1}(B^n(r))$.

Using the harmonic coordinates for g_∞ in $B^n(r)$, we can write the soliton equation as follows:

$$\triangle(g_{ij}) + \cdots = u_{ij} \tag{4.15}$$

where the dots indicate lower order terms involving at most one derivative of g_{ij}. Since $g_\infty \in C^{1,1}(B^n(r))$ and $u_\infty \in C^{1,1}(B^n(r))$, from (4.11) and (4.15), the standard elliptic regularity theory implies that g_∞ and u_∞ must be smooth in $B^n(r)$; that is, g_∞ is a gradient soliton metric in an orbifold sense.

When n is odd, we can use a discussion like that in [1, Section 5] to conclude that there are no curvature singularities in M_∞. We argue by contradiction. Suppose that there exist curvature singularities in M_∞. For each curvature singularity $P \in \{P_i\}_1^Q \subset M_\infty$, there is a sequence $x_\alpha \in M_\alpha$, such that $x_\alpha \to P$ and $\inf_{r>0} \sup\{|\mathrm{Rm}_\alpha(x)|; x \in B_{x_\alpha}(r) \subset M_\alpha\} \to \infty$, as $\alpha \to \infty$. Since the curvature of M_α remains bounded in a bounded distance away from x_α, we may assume that x_α realizes the maximum R_α of $|\mathrm{Rm}_\alpha|$ on $B_{x_\alpha}(r_0)$ for some small r_0. Now consider the pointed connected Riemannian manifolds $V_\alpha = (B_{x_\alpha}, x_\alpha, R_\alpha^{\frac{1}{2}} g_\alpha)$. We note that the curvature of V_α is uniformly bounded, $|\mathrm{Rm}_{V_\alpha}(x_\alpha)| = 1$, and $|\mathrm{Ric}(V_\alpha)|(x) \to 0$ for any point $x \in V_\alpha$ as $\alpha \to \infty$ (since, in Lemma 2.2, we have proved that the Ricci curvature of M_α is bounded uniformly). Similarly, $\int_{V_\alpha} |\mathrm{Rm}|^{\frac{n}{2}} \le C$ and the Sobolev constants for V_α are uniformly bounded below, since this is true for M_α itself. As in section 3 in [1], we can prove that there is a subsequence of V_α that converges, in C^∞ topology on compact sets, to a complete connected Riemannian manifold V satisfying

$$\begin{aligned} &\mathrm{Ric}_V = 0, \\ &\frac{\mathrm{Vol}(B(r))}{r^n} \ge C', \\ &\int_V |\mathrm{Rm}|^{\frac{n}{2}} \le C, \end{aligned} \tag{4.16}$$

and

$$|\mathrm{Rm}|(x_0) = 1, \quad \text{for some } x_0 \in V. \tag{4.17}$$

A complete connected Riemannian manifold satisfying (4.16) is called an EALE (Einstein, asymptotically locally Euclidean) space. Theorem 3.5 in [1] had shown that in odd dimensions, nontrivial, i.e., nonflat, EALE spaces do not exist, so we get the contradiction by (4.17). So it follows that M_∞ is a smooth manifold and the convergence $M_\alpha \to M_\infty$ is smooth.

5. Curvature bounds of the limiting metric

In this section, we will give curvature bounds for the limiting metric g_∞. If $n = 4$, the approach that we will use to prove Lemma 4.2 is based on Uhlenbeck's [22, Theorem 4.1] idea for treating the isolated singularities for the Yang–Mills equation. If $n \ge 5$, using the Sobolev bounds in $B_{g_\infty}(P,r)\backslash\{P\}$, we can verify that the basic methods of Sibner [19, Lemma 2.1, Proposition 2.4] remain valid here also. We should point out that Chao and Sesum [6] had used the same idea for treating the Kähler–Ricci soliton case. We include a sketched proof here.

Let P be a singular point of M_∞, $r(x)=\mathrm{dist}_{g_\infty}(x,P)$, and $B_{g_\infty}(P,r)=\{x\in M_\infty \mid r(x)<r\}$. Since the Sobolev inequality (4.4) with a uniform Sobolev constant C_s' for all g_α, we have the following Sobolev inequality:

$$\left(\int_{B_{g_\infty}(P,r)} f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le \frac{1}{C_s'}\int_{B_{g_\infty}(P,r)} |\nabla f|^2, \tag{5.1}$$

for every Lipschitz function f with compact support in $B_{g_\infty}(P,r)\backslash\{P\}$ and $r\le C_{18}$.

Remark 5.1. By Fatou's lemma and arguments similar to those in [4], we can get that (5.1) also holds for compact supported functions $f\in W^{1,2}(B_{g_\infty}(P,r))$.

From (3.4), we also have

$$\triangle|\mathrm{Rm}|^2 \ge (2-\theta)|\nabla|\mathrm{Rm}||^2 - C(\theta)|\mathrm{Rm}|^2 - C_{14}|\mathrm{Rm}|^3, \tag{5.2}$$

on $M_\infty\backslash\{P_i\}_1^Q$. By (4.9), we can decrease r if necessary so that $\int_{g_\infty(P,r)}|\mathrm{Rm}|^{\frac{n}{2}}\,\mathrm{d}V_{g_\infty}<\epsilon$, where ϵ is chosen to be small. By the Sobolev inequality (5.1) and Lemma 3.1, we have

$$|\mathrm{Rm}(g_\infty)|(x) \le \frac{C}{r(x)^2}\left\{\int_{B_{g_\infty}(P,2r(x))}|\mathrm{Rm}(g_\infty)|^{\frac{n}{2}}\,\mathrm{d}V_{g_\infty}\right\}^{\frac{2}{n}}, \tag{5.3}$$

for some uniform constant C. From Shi's curvature estimates (4.6), letting $\alpha\to\infty$ we get

$$|\nabla^k\mathrm{Rm}(g_\infty)|(x) \le \frac{C(r(x))}{r(x)^{k+2}}, \tag{5.4}$$

for all $x\in M_\infty\backslash\{P_i\}_1^Q$, where $C(r(x))\to 0$ as $r(x)\to 0$.

(a) When $n=4$. Let U be a small neighborhood of P; recall that $U\backslash\{P\}$ is covered by $B^n(r)\backslash\{0\}\subset R^4$ and π^*g_∞ extends to a C^0 metric on the ball $B^n(r)$, where π is the covering map. By estimates (5.3) and (5.4), as in [20, Section 4] we can find a gauge ϕ (i.e., a diffeomorphism on $B^n(r)(r)$) such that

$$\begin{aligned}
&|\mathrm{d}g_{ij}|(x) \le \frac{\epsilon(r(x))}{r(x)},\\
&\left|\mathrm{d}\left(\frac{\partial g_{ij}}{\partial x_k}\right)\right|(x) \le \frac{\epsilon(r(x))}{r(x)^2},\\
&\left|\mathrm{d}\left(\frac{\partial^2 g_{ij}}{\partial x_k\partial x_l}\right)\right|(x) \le \frac{\epsilon(r(x))}{r(x)^3},
\end{aligned} \tag{5.5}$$

in $B^r\backslash\{0\}$, where d is the exterior differential on R^4 and $|\cdot|$ is the norm with respect to the Euclidean metric, and g stands for $\phi^*\pi^*g_\infty$.

Let $D=d+A$ be a connection uniquely associated with the metric g on $B^n(r)\backslash\{0\}$, where A is the connection form. As in [22, Section 4] or [20, Section 4], we constructed the broken Hodge gauges. We break the domain up into annuli:

$$\begin{aligned} \mathbf{U}_l &= \{x : 2^{-l-1}r \leq r(x) \leq 2^{-l}r\}, \\ \mathbf{S}_l &= \{x : r(x) = 2^{-l}r\}, \end{aligned} \tag{5.6}$$

for $l = 0, 1, 2, \ldots$.

Definition 5.2. A broken Hodge gauge for a connection D in a bundle E over $\cup_{l=0}^{\infty} \mathbf{U}_l$ is a gauge related continuously to the original gauge in which $D = \mathrm{d} + A$ and $A(l) = A|_{\mathbf{U}_l}$ have the following properties for all l:

$$\begin{aligned} &d^* A(l) = 0 \quad \text{in } \mathbf{U}_l, \\ &A_\psi(l)|_{\mathbf{S}_l} = A_\psi(l-1)|_{S_l}, \\ &d^*_\psi A_\psi(l) = 0 \quad \text{on } \mathbf{S}_l \text{ and } \mathbf{S}_{l+1}, \\ &\int_{\mathbf{S}_l} A_r(l) = \int_{\mathbf{S}_{l+1}} A_r(l) = 0. \end{aligned}$$

As in [22, Theorem 4.6] or [20, Section 4], from estimate (5.3), we have the following lemma.

Lemma 5.3. *Let D be the unique connection associated with the metric g, for small r; then there exists a broken Hodge gauge in $B^n(r)\backslash\{0\} = \cup_{l=0}^{\infty} \mathbf{U}_l$ satisfying*

$$\begin{aligned} &|A(l)|_g(x) \leq C2^{-l} r \sup_{\mathbf{U}_l} |\mathrm{Rm}|_g \leq C2^{l+1} r^{-1}, \\ &\int_{\mathbf{U}_l} |A(l)|_g^2 \,\mathrm{d}V_g \leq C2^{-2l} r^2 \int_{\mathbf{U}_l} |\mathrm{Rm}|_g^2 \,\mathrm{d}V_g. \end{aligned} \tag{5.7}$$

By direct calculation, we have

$$\begin{aligned} \sum_{l=0}^{\infty} \int_{\mathbf{U}_l} |\mathrm{Rm}|_g^2 \,\mathrm{d}V_g = &-\sum_{l=0}^{\infty} \int_{\mathbf{U}_l} \langle A(l), D^*\mathrm{Rm}\rangle - \sum_{l=0}^{\infty} \int_{\mathbf{U}_l} \langle [A(l), A(l)], \mathrm{Rm}\rangle \\ &+ \int_{\mathbf{S}_0} \langle A_\psi(0), \mathrm{Rm}_{r\psi}\rangle - \lim_{l\to\infty} \int_{\mathbf{S}_{l+1}} \langle A_\psi(l), \mathrm{Rm}_{r\psi}\rangle. \end{aligned} \tag{5.8}$$

From (5.3) and (5.7), it is not hard to check that $\lim_{l\to\infty} \int_{\mathbf{S}_{l+1}} \langle A_\psi(l), \mathrm{Rm}_{r\psi}\rangle = 0$. On the other hand, we know that the limit metric g_∞ satisfies the Ricci soliton equation

$$g - \mathrm{Ric} = \nabla \mathrm{d}u, \tag{5.9}$$

where $u = \phi^*\pi^* u_\infty$. We have

$$D^*\mathrm{Rm}_{ijk} = R_{ijkm,m} = R_{ki,j} - R_{kj,i} = u_{k,ji} - u_{k,ij} = R_{ijkl} g^{lm} u_m \tag{5.10}$$

where we have used the second Bianchi identity and the Ricci identity. By Lemma 2.2, we know that $|\nabla u|$ is bounded uniformly, so we have

$$\int_{\mathbf{U}_l} \langle A(l), D^*\mathrm{Rm}\rangle \leq \left(\int_{\mathbf{U}_l} |A(l)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbf{U}_l} |D^*\mathrm{Rm}|^2\right)^{\frac{1}{2}} \leq C2^{-l} r \int_{\mathbf{U}_l} |\mathrm{Rm}|_g^2. \tag{5.11}$$

From the estimate (5.3), we get

$$\left|\int_{\mathbf{U}_l}\langle[A(l),A(l)],\mathrm{Rm}\rangle\right| \le \sup_{\mathbf{U}_l}|\mathrm{Rm}|_g\int_{\mathbf{U}_l}|A(l)|^2 \le C\left(\int_{B_{g_\infty}(P,2^{-l+1}r)}|\mathrm{Rm}(g_\infty)|^2\right)^{\frac{1}{2}}\int_{\mathbf{U}_l}|\mathrm{Rm}|_g^2. \tag{5.12}$$

Similarly to in the proof of Corollary 2.6 in [22], with a small modification just like that in [20, Section 4], we can find a decreasing function $\epsilon_1(r)$ with $\lim_{r\to 0}\epsilon_1(r)=0$ such that

$$\int_{\mathbf{S}_0}|A_\psi(0)|_g^2\,\mathrm{d}V_g \le (2-\epsilon(r))^{-2}r^2\int_{\mathbf{S}_0}|\mathrm{Rm}_{\psi\psi}|_g^2\mathrm{d}V_g. \tag{5.13}$$

From (5.8) and (5.11)–(5.13), we have

$$\int_{B(r)}|\mathrm{Rm}|_g^2\,\mathrm{d}V_g \le \frac{r}{2(2-\epsilon_1(r))(1-\epsilon_2(r))}\int_{\partial B(r)}|\mathrm{Rm}|_g^2\,\mathrm{d}V_g \le \frac{r}{4}\left(1+\frac{\delta}{2}\right)\int_{\partial B(r)}|\mathrm{Rm}|_g^2\,\mathrm{d}V_g, \tag{5.14}$$

whenever r is sufficiently small and $\delta\in(0,1)$, where $\lim_{r\to 0}\epsilon_2(r)=0$. Then it is standard to conclude from the above inequality that [20, Section 4]

$$|\mathrm{Rm}|_{g_\infty}(x)\le\frac{C}{r(x)^\delta}, \tag{5.15}$$

for $x\in B_{g_\infty}(P,r)$, for sufficiently small r and some $\delta\in(0,1)$, where C is a uniform constant. Recall that g_∞ extends the C^0 metric on the covering ball; from (5.15), we can show that there exists a $q>4$ such that

$$\int_{B_{g_\infty}(P,r)}|\mathrm{Rm}(g_\infty)|^q<\infty. \tag{5.16}$$

For further consideration, we need the following lemma which is similar to Lemma 2.1 in [19], and had been proved in [6].

Lemma 5.4. *Let $f\ge 0$ be a smooth function in $M_\infty\backslash\{P_i\}_1^Q$ and satisfying (3.5), with $f\in L^{\frac{n}{2}}$. If $f\in L^{\frac{2nq_0}{n-2}}\cap L^{2q}$, $q_0>\frac{1}{2}$, then $\nabla f^q\in L^2$ and, for sufficiently small r, we have*

$$\int_{B_{g_\infty}(P_i,r)}\eta^2|\nabla f^q|^2\le\int_{B_{g_\infty}(P_i,r)}|\nabla\eta|^2 f^{2q}, \tag{5.17}$$

for all $\eta\in B_{g_\infty}(P_i,r)$, where C is a uniform constant.

(b) If $n>4$, let $f=|\mathrm{Rm}(g_\infty)|\in L^{\frac{n}{2}}$; we can choose $q_0=1$ and $q=\frac{n}{4}$. From (5.2), we know that $f=|\mathrm{Rm}(g_\infty)|\in L^{\frac{n}{2}}$ satisfies (3.5); applying Lemma 5.4 to f, we find that $\nabla f^{\frac{n}{4}}\in L^2$. By Remark 5.1, we can apply the Sobolev inequality (5.1) to $f^{\frac{n}{4}}$ to conclude that

$$|\mathrm{Rm}(g_\infty)|\in L^p, \tag{5.18}$$

with $p=\frac{n}{2}\frac{n}{n-2}>\frac{n}{2}$.

From the above we know that $|\text{Rm}(g_\infty)| \in L^p$ for some $p > \frac{n}{2}$ in both the cases $n = 4$ and $n > 4$. In particular, since $\text{Vol}(M_\infty, g_\infty) < \infty$, by (5.16), (5.18) and using the Hölder inequality, we have $|\text{Rm}(g_\infty)| \in L^p$ for $p \in (0, \frac{n}{2}\frac{n}{n-2}]$. Take $q_0 = 1$, $q \in (0, \frac{n}{4}\frac{n}{n-2}]$ and repeatedly apply Lemma 5.4 to get $\nabla|\text{Rm}_{g_\infty}|^q \in L^2$. By the Sobolev inequality (5.1) and using the Hölder inequality again, we have $|\text{Rm}(g_\infty)| \in L^p$ for $p \in (0, \frac{n}{2}(\frac{n}{n-2})^2]$. If we keep on repeating this, at the k-th step we have $\nabla|\text{Rm}(g_\infty)|^q \in L^2$ for $q \in (0, \frac{n}{4}(\frac{n}{n-2})^k]$ and $|\text{Rm}(g_\infty)| \in L^p$ for $p \in (0, \frac{n}{2}(\frac{n}{n-2})^{k+1}]$. Since $(\frac{n}{n-2})^k \to \infty$ as $k \to \infty$, we have

$$|\text{Rm}(g_\infty)| \in L^p, \tag{5.19}$$

and

$$\nabla|\text{Rm}(g_\infty)|^p \in L^2, \quad \text{for all } p. \tag{5.20}$$

By (5.19) and (5.20), combining Remark 5.1 and Lemma 5.4, for sufficiently small r and any p we have

$$\begin{aligned}\left(\int_{B_{g_\infty}(P,r)} |\eta|\text{Rm}_{g_\infty}|^p|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\le \frac{1}{C_s'}\int_{B_{g_\infty}(P,r)} |\nabla(\eta|\text{Rm}(g_\infty)|^p)|^2 \\ &\le C\int_{B_{g_\infty}(P,r)} |\nabla\eta|^2|\text{Rm}(g_\infty)|^{2p},\end{aligned} \tag{5.21}$$

with a uniform constant C, where η is any cut-off function with compact support in $B_{g_\infty}(P, r)$. Then, using Moser's iteration argument as in the proof of Lemma 3.1, we get

$$\sup_{B_{g_\infty}(P,\frac{r}{2})} |\text{Rm}(g_\infty)| \le \frac{C}{r^2}. \tag{5.22}$$

So, we get the curvature bound for the limiting metric g_∞ and we have finished the proof of Lemma 4.2.

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